A Decomposition Scheme in Production Planning Based on Linear Programming that Incorporates the Concept of a Dynamic Planning Environment

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Abstract. The present work proposes a decomposition scheme in production planning based on linear programming, which couples the concept of clearing function and supposes a dynamic environment to plan a production along a planning horizon. Where planning parameters such as demand, production capacity, production resources, and other parameters change over the planning horizon. In general, this requires the planner, freeze the production plan for some blocks of periods or rethink the production plan of each period, with obvious implications on production planning costs. The Plan shall be designed and implemented in the following way: At the beginning of any period of the planning horizon, from the first, a set of decisions on a production is taken based on available information under a supervision of a decision support system represented by the mathematical model that acts according to the latest available information, period to period. This mechanism suggest that the number of planned periods is of the magnitude of n squared, much larger than the number of periods to be performed, because for each execution period, the remainder of the planning horizon is re-planned. The scheme we are proposing in this paper addresses this drawback and provides a lower cost solution to this production planning problem. This proposed scheme is implemented by an algorithm which is analyzed in detail and after, numerically illustrated.

Key words: Production Planning, Clearing Function, Linear Programming.

1 Introduction

Currently, manufacturing companies are increasingly demanding that production planning be supported by effective and efficient decision making systems so that they can achieve their strategic objectives using the available resources in the best possible way.

Given this scenario, a large number of medium and large manufacturing companies have been implementing sophisticated decision support systems, such as Master Production Schedule (MPS), which aims to establish which end products will be manufactured in a given period of time and in what quantity; Material Requirement Planning (MRP), which allows that based on the production decisions of the final products it is determined what, when, and how much to produce [1] and Manufacturing Resource Planning (MRP II). However, these models don't even have any objective action in face of eventual capacity limitations, besides that, these models only look for a viable solution, without any concern with the quality of the solution found, because it indexes the quality of the solution based on stock and Lead Time and more importantly, without considering production costs.

As a consequence of the deficiencies of these models in dealing with capacity constraints, since the 1950s, models that seek optimal solutions such as Linear Programming (LP) models have been widely used to solve production planning problems. And these have been studied by several authors over several decades [2]; [3];[4]; [5].

However, production planning problems require constant updates, mainly due to the environment in which they are inserted and even the responsible planner. Due to this, that is, the new information that is fed into the model, the planner is forced to replan the rest of the planning horizon, and this, in any production process, represents a waste of resources and generates nervousness in the planning and in the system. This is particularly severe when the planning horizon is long and the environment is dynamic, that is, there is a greater uncertainty in the input parameters.

According to Wollmann [6], these aspects of production planning, which require sequential decisions over time, can be referred to as a Rolling Planning approach. The Rolling Planning rule works when the planning horizon is fixed with a fixed number of periods ahead; the first production planning is formulated covering all periods of the planning horizon; production planning is performed only on the part concerning the first period, this process is performed repeatedly until the number of periods runs out [5]. Therefore, each of the production planning horizons is a shorter period than the previous one, until it ends, and, in this sense, it is a decomposition process. Within this context, the present work proposes a method for decomposing the mathematical model of production planning based on linear programming that couples the concept of dynamic environment to plan production over a production planning horizon, where planning parameters such as demand, production capacity, production costs, among other parameters, vary over the planning horizon.

2 Theoretical Background

2.1 Production Planning Models

According to Gaither [7], MRP is a computerized system that takes the Master Production Schedule (MPS) as data, explodes it into the quantity required for each period in the planning horizon, reduces it by comparing it with the respective stocks or receipts, and thus develops an order schedule. The same author states that "the outputs of the MRP systems dynamically provide the materials program for the future - the quantity of each material needed in each period to support the MPS."

It is important to emphasize that MRP starts from the vision of the future of the need for products and then comes "exploding" backwards according to lead time, level by level, the needs of components. Therefore, this system is known as the "backward scheduling" logic ([8]; [9];[3]; [10];[11]).

Baker [8], defines that the purpose of the MRP logic is to try to answer the question, "What do we need and when?" In order to consider this question, the goal is to treat the process systematically, determining purchasing and manufacturing plans based on net requirements. This mechanics is done by sequences of steps, called: Explosion; Network adjustment; Lead time and lot size.

However, MRP has some problems in its system, some of these weaknesses need to be pointed out for comparison of the studied tools, which support the proposed model, they are: (i) Unviability of the MRP scheduling capacity, (ii) Lead times planned for a long period of time, (iii) System instability ([8]; [3];[4]).

With the drawbacks presented by the MRP system, over time new procedures were emerging and with the growth of technology employed in computers came the emergence of the MRP II, this, adding a possibility of processing and more agile communication between more sectors of the organization. The MRP II is based on an integrated system, having an important utility in the midst of demand changes [11]. It has allowed to explore any change that an operation needs to make.

However, even with the emergence of MRP II, this new model did not solve the problems faced with eventual capacity limitations detected. Because of this, since 1950, mathematical programming models have been widely used to solve production planning problems, which will be addressed next.

2.2 Linear Programming

Linear Programming (LP) is a mathematical programming in which the objective function and the restrictions assume linear characteristics, having several applications in management control, including problems of production, mixing, transportation, determination of inventory policy, cash flow studies, study of information system among others, in summary, problems of use of available resources that seek optimal use of them, observing limitations imposed by the production process or by the market. For [12] linear programming is used when the objective is to solve problems that consider the optimal allocation of scarce resources throughout production or in the performance of activities.

As previously mentioned, this work treats in a generalized way the production problem addressed in [5], where a production planning scheme, initially proposed by [1], and complemented by [13], in which he proposed a mathematical model for production planning, using decomposition, considering the functional constraints as integers, moreover, relaxing the integrality constraint. In this approach, according to [6], the classical LP (1) model for production planning problems can be formulated as a sparse structured linear problem, whose structure is quite suitable for decomposition, and which describe cumulatively the resources whose leftovers can be transferred from one period to the next, we consider the following production planning problem: Let X_{ij} be the production level of product i in period j. Let b_{ki} be the number of components k used to produce one unit of product i. $R_{j,}$is the amount of labor resource (in units of standard time) available during period j, and that any unused labor resource from period j cannot be carried out to period j +1. Let S_{kj}be the supply of components k available for consumption in period j, and let D_i, be the maximum demand for product i until the end of the planning horizon.

$$Maximize \sum_{j=1}^{T}\sum_{i=1}^{N} C_{ij}\, X_{ij}$$

s.t.

$$\begin{aligned} \sum_{j=1}^{t}\sum_{i=1}^{I} b_{ki}X_{ij} \ &\leq \ \sum_{j=1}^{t} S_{kj} \quad \forall k, \forall t, \\ \sum_{i=1}^{N} h_i X_{ij} \leq \ & R_{j,} \forall j, \\ X_{ij} \ &\leq \ C_{ij}, \forall i, \forall j, \\ \sum_{j=1}^{T} X_{ij} &\leq D_i, \forall i. \\ X_{ij} &\geq 0, \forall i, \forall j \end{aligned} \tag{1}$$

In continuation, [6] added the lead time issue to this model, to consider the variability that exists in a production system, and to consider the lead time it was necessary to introduce the Clearing Function (CF) concept to the model. To complement this approach, [5] added the Rolling Planning rule to this problem, with the purpose of avoiding rework and unnecessary costs with production planning throughout the planning horizon, being able to consider period to period, allowing for the updating of model parameter information.

3 Proposed Model

In this scenario, this paper proposes to continue this exposed problem, adding to the model the decision variables regarding the level of inventory, with values of costs of handling high inventories, to highlight the loss of income in the period that there is inventory, as well as the restrictions associated with the inventory balance.

3.1 Sets and Indices

The proposed model is described below (1):
Index:
j – Indicates the period $j = 1,2, \dots, T$.
i – Indicates the product $i = 1,2, \dots, N$.
k – Indicates the component $k = 1,2, \dots, K$.

3.2 Decisions Variables

X_{ij} *– Production level of product i on period j;*
I_{ij} *– Inventory level of product i on period j;*

3.3 Parameters

c_{ij} *– Unit contribution of product production i on period j;*
h_{ij} *– Unit cost of product inventory i on period j;*
γ_{ij} *– Nominal production capacity, estimated by CF, of the product i period j;*
D_i *– Product Demand i;*
d_i *– Minimum product demand i period j;*
b_{ki} *– Number of components k used for the production of the product of a i.*
S_{kj} *– Number of components of type k, available to be consumed at the beginning of j;*

R_j *– Available labor (in standard units of time) to be used in j;*
h_i *– Standard time required to produce one unit of the product i.*

3.4 Objective Function:

$$\sum_{j=1}^{J}\sum_{i=1}^{I}\{(X_{ij}+I_{ij-1})c_{ij}-c_{ij}I_{ij}-\ h_{ij}I_{ij}\};$$

The function evaluates the yields in relation to the production level and what has been kept in stock, i.e., turnover accounts for products sold and those still in stock.

3.5 Restrictions:

1- Restriction of the components of each product:

$$\sum_{j=1}^{t}\sum_{i=1}^{I}b_{ki}X_{ij}\leq\sum_{j=1}^{t}S_{kj}\qquad,\forall k,\forall t$$

It ensures that the component quantity used in the production of each product is less than or equal to the total material resources available in that period.

2- Inventory Balancing Restriction:

$$I_{i,j}=I_{i,j-1}+X_{i,j}-D_{i,j}\qquad,\forall i,\forall j$$

It ensures the balance between stock levels and production to meet what is demanded in each period.

3- Capacity restriction

$$X_{ij}-\ \gamma_{ij}\leq 0\qquad,\forall i,\forall j$$

It guarantees that the production of product i period j is according to the available capacity in that period, estimated by CF.

The capacity parameter γ_{ij} of this model is governed by the Clearing Function concept [4]. Recently several authors, have proposed models that use CF concepts, such as [13] and more recently by [6];[14];[5];[15]). As discussed in [4] and in most of the studies cited above, it is assumed that the CF is an increasing concave function with decreasing rate, which when coupled to the Linear Programming turns the model a convex programming model. The scholars who approach this model use some resources to linearize the CF by means of affine straight lines in order to obtain a LP model. Since this mathematical programming model used considers CF as a model parameter, no further discussion on this subject will be dealt with in this paper, only considerations already mentioned. An application considering other service levels can be seen in ([19];[20];[21]).

4 – Demand Constraints

$$\sum_{j=1}^{T}X_{ij}\leq D_i\qquad,\forall i,\forall t$$

Demand is an external parameter to the model, and since we have a maximization model, this constraint prevents all production from being done in a single period, ensuring the model's feasibility.

5 – Minimum Demand Constraints

$$X_{ij}\ \geq\ d_{ij}\qquad,\forall i,\forall j$$

It also guarantees the feasibility of the model by having a minimum production to be realized.

6- Restriction of Labor Resources

$$\sum_{i=1}^{I}h_iX_{ij}\leq\ R_j\qquad,\forall i,\forall j$$

It ensures that all the available labor resource is used throughout the period.

Thus, the proposed linear programming model is presented below (2):

$$Maximize\sum_{j=1}^{J}\sum_{i=1}^{I}\{(X_{ij}+I_{ij-1})c_{ij}-c_{ij}I_{ij}\\-\ h_{ij}I_{ij}\}$$

$$s.a \qquad \sum_{j=1}^{t}\sum_{i=1}^{I} b_{ki}X_{ij} \le \sum_{j=1}^{t} S_{kj} \qquad \forall k, \forall t$$

$$I_{i,j} = I_{i,j-1} + X_{i,j} - D_{i,j} \qquad \forall i, \forall j$$

$$X_{ij} - \gamma_{ij} \le 0 \qquad \forall i, \forall j \qquad (2)$$

$$\sum_{j=1}^{T} X_{ij} \le D_i \qquad \forall i, \forall t$$

$$X_{ij} \ge d_{ij} \qquad \forall i, \forall j$$

$$\sum_{i=1}^{I} h_i X_{ij} \le R_j \qquad \forall i, \forall j$$

$$X_{ij} \ge 0, I_{ij} \ge 0, I_{i0} = 0.$$

4. Decomposition of the Proposed Model

The decomposition scheme of the proposed mathematical model (2) ensures that all planning horizons have the subproblems as sub horizons of the entire planning horizon and use the current information available in each period of the planning horizon. In order to discuss the decomposition of model (2) it is necessary to define the representation of the following vectors, for each period of the planning horizon.

(1) $x^{(j)} = \left(x_{1j}, x_{2j}, \dots, x_{Nj}\right)^T, \quad j = 1, 2, \dots, T;$
(2) $B = (b_{ki}), \quad k = 1, 2, \dots, K, \quad i = 1, 2, \dots, N;$
(3) $S^{(j)} = \left(S_{1j}, S_{2j}, \dots, S_{Kj}\right)^T, \quad j = 1, 2, \dots, T;$
(4) $R^{(j)} = R_j, \quad j = 1, 2, \dots, T;$
(5) $D^{(j)} = (D_1, D_2, \dots, D_N)^T;$
(6) $c^{(j)} = \left(c_{1j}, c_{2j}, \dots, c_{Nj}\right), \quad j = 1, 2, \dots, T;$
(7) $h = (h_1, h_2, \dots, h_N);$
(8) $\gamma^{(j)} = \left(\gamma_{1j}, \gamma_{2j}, \dots, \gamma_{Nj}\right), \; j = 1, 2, \dots, T;$
(9) $\theta^{(j)} = \left(\theta_{1j}, \theta_{2j}, \dots, \theta_{Nj}\right), \; j = 1, 2, \dots, T;$ (Regarding the labor force maintenance vector)
(10) I, Matrix Identity (relative to size).

Model (2) couples the Clearing Function with the linear programming model to account for the nonlinear variability of production throughput with workload and system runtimes. Since the Clearing Function is defined for each resource in each period of the planning horizon, it can be combined with the decomposition scheme of the mathematical model. However, in this paper the Clearing Function is not discussed further.

The decomposition scheme follows the following process and ensures the rolling planning scheme defined here: assuming that the planning horizon T is divided into M blocks of P periods, that is, $T = M$ x P. Then, the Matrix $\mathbb{A}$ block of the model variables (2) and the right-hand side vectors, referring to resources, $\mathbb{F}$ as can be seen below (3) is defined. Where each block of the matrix, B, I, h, is respectively the matrix $B_{K \times N}$, the matrix $I_{N \times N}$, , a vector $h_{1 \times N}$,, and a vector 1 is the sum of the vector. As a result, the matrix $\mathbb{A}$ is an $\mathbb{A}_{[(K + N + 1 + 1) \times \mathrm{T}] \times [(\mathrm{T} \times \mathrm{N})]}$ matrix, in which it is then decomposed into M blocks of size P, and according to the vector $\mathbb{F}$.

$$\mathbb{A} = \begin{pmatrix} B & & & & \\ I & & & & \\ h & & & & \\ \mathbf{1} & & & & \\ B & B & & & \\ I & I & & & \\ 0 & h & & & \\ 0 & \mathbf{1} & & & \\ \vdots & & & & \\ B & B & B & \dots & B \\ I & I & I & \dots & I \\ 0 & 0 & 0 & & h \\ 0 & 0 & 0 & & \mathbf{1} \end{pmatrix}, \quad \mathbb{F} = \begin{pmatrix} S^{(1)} \\ D^{(1)} \\ R^{(1)} \\ \gamma^{(1)} \\ S^{(1)} + S^{(2)} \\ D^{(2)} \\ 0 + R^{(2)} \\ 0 + \gamma^{(2)} \\ \vdots \\ S^{(1)} + S^{(2)} + \dots + S^{(M)} \\ D^{(M)} \\ 0 + 0 + \dots + R^{(M)} \\ 0 + 0 + \dots + \gamma^{(M)} \end{pmatrix} \tag{3}$$

Each block of matrix $\mathbb{A}$ will be numbered as $\mathbb{A}^{(j)}$, and the right hand side the vector $\mathbb{F}$, as $\mathbb{F}^{(j)}$. Since $\mathbb{A}^{(j)}$ is fixed, however the size of the blocks is held constant, this will be referred to as matrix $\mathbb{G}$. The defined vectors are:

$$X^{(1)} = \begin{pmatrix} x^{(1)} \\ x^{(2)} \\ \vdots \\ x^{(P)} \end{pmatrix};\ X^{(2)} = \begin{pmatrix} x^{(P+1)} \\ x^{(P+2)} \\ \vdots \\ x^{(2P)} \end{pmatrix};\ \cdots;\ X^{(M)} = \begin{pmatrix} x^{(MP-P+\ )} \\ x^{(MP-P+2)} \\ \vdots \\ x^{(MP)} \end{pmatrix}, \tag{4}$$

$$C^{(1)} = \begin{pmatrix} c^{(1)} \\ c^{(2)} \\ \vdots \\ c^{(P)} \end{pmatrix};\ C^{(2)} = \begin{pmatrix} c^{(P+1)} \\ c^{(P+2)} \\ \vdots \\ c^{(2P)} \end{pmatrix};\ \cdots;\ C^{(M)} = \begin{pmatrix} c^{(MP-P+1)} \\ c^{(MP-P+2)} \\ \vdots \\ c^{(MP)} \end{pmatrix}. \tag{5}$$

The data update vector ${\mathbb{L}_{\mathbb{NC}}}^{(j)} = D^{(j)} - \sum_{t=j+1}^{T} \gamma^{(t)}, j = 1, \dots, M-1, \mathbb{L}^{(M)} = 0$, where ${\mathbb{L}_{\mathbb{NC}}}^{(j)}$represents the smallest output per period j, and the update vector of dados ${\mathbb{L}_{\mathbb{MO}}}^{(j)} \longleftarrow h^T D^{(j)} - h^T \sum_{t=j+1}^{T} \theta^{(t)}$, where ${\mathbb{L}_{\mathbb{MO}}}^{(j)}$ represents the labor resource for period j, in which both the production and labor resource vector ensures feasibility for the problem. $D^{(j)}$ is the maximum attainable demand of the planning horizon, i.e., the demand from period j to the end of the planning horizon in a cumulative manner. Considering the decomposition of the proposed model the following algorithm was defined.

Algorithm

Initialization

Solve the subproblem.

$$\begin{aligned} & \quad Maximize\ \ C^{(1)} X^{(1)} \\ s.\ t. \quad & \mathbb{G}\, X^{(1)} \le \mathbb{F}^{(1)} \\ & X^{(1)} \ge {\mathbb{L}_{\mathbb{NC}}}^{(1)} \\ & X^{(1)} \ge {\mathbb{L}_{\mathbb{MO}}}^{(1)} \end{aligned} \tag{6}$$

Iteration

TO $j = 2, \dots, M$, upgrade the vectors $\mathbb{F}^{(j)}$, ${\mathbb{L}_{\mathbb{NC}}}^{(j)}$ e ${\mathbb{L}_{\mathbb{MO}}}^{(j)}$ as follows,

$$\mathbb{F}^{(j)} \longleftarrow \mathbb{F}^{(j)} + \left[\mathbb{F}^{(j-1)} - \mathbb{G}\hat{X}^{(j-1)}\right], \tag{7}$$

$${\mathbb{L}_{\mathbb{NC}}}^{(j)} \longleftarrow D^{(j)} - \textstyle\sum_{t=j+1}^{T} \gamma^{(t)},$$

$${\mathbb{L}_{\mathbb{MO}}}^{(j)} \longleftarrow h^T D^{(j)} - h^T \textstyle\sum_{t=j}^{T} \theta^{(t)},$$

$$D^{(j)} \longleftarrow \left[D^{(j)} - \hat{X}^{(j-1)}\right],\ \left(D^{(1)} = D\right),\ D^{(j)} = D^{(1)} - \textstyle\sum_{i=1}^{j-1} \hat{X}^{(i)}$$

And solve the subproblem,

$$\begin{aligned} &Maximize \;\; C^{(j)}X^{(j)} \\ s.\ t. \quad &\mathbb{G}\, X^{(j)} \leq \mathbb{F}^{(j)} \\ &X^{(j)} \geq \mathbb{L}_{\mathbb{NC}}{}^{(j)} \\ &X^{(j)} \geq \mathbb{L}_{\mathbb{MO}}{}^{(j)} \end{aligned} \tag{8}$$

$\hat{X}^{(j)}$ represents an optimal solution for $j^{(esimo)}$ subproblem $j = 1,..,M.$

The following theorem proves that the algorithm presented above implements a rolling-planning rule that spans a fixed planning horizon that reduces the planning horizon by one period, each period of its execution.

Theorem. Assume that we have a fixed planning horizon of T periods, and a production plan for that planning horizon. Assume further that the plan must be remade at the end of each period until the end of the planning horizon, in other words, throughout the planning horizon production is planned to cover the entire planning horizon, but executed only for the next period, until the end of the planning horizon. Assuming that all parameters of the mathematical model (2) are known, but, can be varied from one period to another and that $X^{*} = \left(X^{*(1)}, X^{*(2)}, \ldots, X^{*(M)}\right)$ is an optimal solution to problem (1) and that has sufficient resources to produce and meet the estimated demand at the end of the planning horizon. Therefore, algorithm (6) and (7) is defined correctly, and if $\widehat{X}^{(j)},\ j = 1, 2, \ldots, M$ are optimal solutions to the subproblems M. The combination of vectors of the solutions $\widehat{X} = \left(\hat{X}^{(1)}, \hat{X}^{(2)}, \ldots, \hat{X}^{(M)}\right)$, is also considered an optimal solution to problem (1).

Proof.

To prove the statement of the above theorem, first, observe the problem (1) and the set of subproblems defined by Algorithm (6) and (7) have the same set of information, the same set of variables, and the same schedule. Therefore, it is reasonable to assume that they have the same set of solutions.

For the purpose of carrying out the formal proof, mathematical induction has been used. First, it will be proved that the theorem is true for periods of T = 2, and further, assuming the result is true for periods of T = k, it will be proved that it is also true for periods of T = k + 1, therefore, concluding the result is true for any natural number of periods T.

Proof for T=2 periods.

Suppose that $X^{*} = (X^{*(1)},\ X^{*(2)})$ is an optimal solution to problem (P),

$$\begin{aligned} &Maximize \;\; C^{(1)}X^{(1)} + C^{(2)}X^{(2)} \\ s.t. \quad &\mathbb{G}X^{(1)} \leq \mathbb{F}^{(1)}, \\ &\mathbb{G}X^{(1)} + \mathbb{G}X^{(2)} \leq \mathbb{F}^{(1)} + \mathbb{F}^{(2)}, \\ &X^{(1)},\ X^{(2)} \geq 0. \end{aligned} \tag{P}$$

Problem (P) is problem (1) for $T = 2$ periods, or two period blocks. For this, the following statements are true:

$$\mathbb{G}X^{*(1)} \leq \mathbb{F}^{(1)} \text{ , e } 0 \leq X^{*(1)} \leq \gamma^{(1)};$$

$$\mathbb{G}X^{*(1)} + \mathbb{G}X^{*(2)} \leq \mathbb{F}^{(1)} + \mathbb{F}^{(2)},\ 0 \leq X^{*(2)} \leq \gamma^{(2)}, \text{e} X^{*(1)} + X^{*(2)} \leq \gamma^{(1)} + \gamma^{(2)},$$

$$C^{(1)}X^{(1)} + C^{(2)}X^{(2)} \leq C^{(1)}X^{*(1)} + C^{(2)}X^{*(2)} \leq C^{(1)}\gamma^{(1)} + C^{(2)}\gamma^{(2)}, \text{ for any feasible solutionl}$$

$$X = \left(X^{(1)}, X^{(2)}\right).$$

Now suppose that $\hat{X}^{(1)}$ is an optimal solution to the first subproblem, defined by Algorithm (6) and (7). Obviously, $\hat{X}^{(1)} \leq X^{*(1)} + X^{*(2)}$ and in addition:

$$\mathbb{G}\hat{X}^{(1)} \leq \mathbb{F}^{(1)}, 0 \leq \hat{X}^{(1)} \leq \gamma^{(1)},$$

$$\hat{X}^{(1)} + \gamma^{(2)} \geq D \text{ , e}$$

$$h^{T}\hat{X}^{(1)} + h^{T}\theta^{(2)} \;\geq\; \mathbb{L}_{\mathbb{MO}}{}^{(2)}$$

Therefore, if $\hat{X}^{(2)}$ is an optimal solution to the second problem, defined by Algorithm (6) and (7),

$$\mathbb{G}\hat{X}^{(2)} \leq \mathbb{F}^{(2)} + \left(\mathbb{F}^{(1)} - \mathbb{G}\hat{X}^{(1)}\right) , 0 \leq \hat{X}^{(2)} \leq \gamma^{(2)},$$

$\hat{X} = (\hat{X}^{(1)}, \hat{X}^{(2)})$ is a feasible solution to problem (P), and,

$C^{(2)}X^{(2)} \leq C^{(2)}\hat{X}^{(2)} \leq C^{(2)}\gamma^{(2)}$, for all feasible solutions to the second subproblem.

Combining the above results and considering the algorithm, we will have:

$$\hat{X}^{(1)} + \gamma^{(2)} \geq D, \text{ and}$$
$$\hat{X}^{(2)} \leq \gamma^{(2)}.$$

So, or both $\hat{X}^{(2)} = \gamma^{(2)}$, and this result:

$$C^{(1)}\hat{X}^{(1)} + C^{(2)}\hat{X}^{(2)} \geq C^{(1)}X^{*(1)} + C^{(2)}X^{*(2)},$$
or
$\hat{X}^{(2)} \leq \gamma^{(2)}$ and, in this case, $\hat{X}^{(1)} + \hat{X}^{(2)} = D$, and consequently,
$$C^{(1)}\hat{X}^{(1)} + C^{(2)}\hat{X}^{(2)} \geq C^{(1)}X^{*(1)} + C^{(2)}X^{*(2)}.$$

Therefore, in any case, $\hat{X} = (\hat{X}^{(1)},\ \hat{X}^{(2)})$, is also an optimal solution to the problem, and the theorem is true for $T = 2$ periods.

Proof for any T periods.

Now assume that the theorem is true for $T = k$ periods and assume that it is for $T = k + 1$ periods.

Since it has been proved that the result for $T = 2$ periods, considering a set of new variables $\bar{X} = \left(\bar{X}^{(1)}, \bar{X}^{(k+1)}\right)$ where $\bar{X}^{(1)} = \left(X^{(1)},\ X^{(2)}, \ldots X^{(k)}\right)$ and according to a new cost vector $\bar{C}^{(1)} = \left(\bar{C}^{(1)}, \bar{C}^{(k+1)}\right)$. Therefore, the resulting problem is the two-period problem above, and thus the result is true for $\bar{X}$, which is equivalent to periods $T = k + 1$periods. Which assures us that the proof is complete.

Corollary.

Suppose the assumptions of the above theorem are true, that the planning horizon is fixed and divided into several periods, and that rolling planning, where one plans the entire planning horizon and only executes the plan through the first period to the end of the planning horizon is in order, then the mathematical model decomposition scheme provided by Algorithm (6) and (7) performs exactly this rolling planning rule.

This corollary is obvious, but worth mentioning since it performs the role of rolling planning by just planning the period that will be executed.

5. Illustrative Numerical Experiment

In the example illustrated below, the solutions to the production planning problem solved for the entire production planning horizon are shown without the proposed decomposition scheme and with the decomposition of the mathematical model, which has the same input parameters. For numerical illustration of the proposed model, we considered a production system with two products, each with different components and a planning horizon of 12 periods. Random values were assigned to the model's production parameters and the solutions obtained were generated using the Lingo 15.0.3 software on a Dell - core i5 machine and the analysis was done in the Excel software, where the tables are also presented.

5.1 Data

The following tables present the model input data, covering demand per period, the costs associated with production, inventory, and production capacity for the problem without decomposition.

Table 1. Production yield parameters

Product 1	**c11**	**c12**	**c13**	**c14**	**c15**	**c16**	**c17**	**c18**	**c19**	**c110**	**c111**	**c112**
	26	25	30	32	36	32	37	16	20	32	26	28
Product 2	**c21**	**c22**	**c23**	**c24**	**c25**	**c26**	**c27**	**c28**	**c29**	**c210**	**c211**	**c212**
	22	28	28	30	40	34	40	26	27	30	27	25

Table 3. Production components

Product components 1	**b11**	**b12**
	2	3
Product components 2	**b21**	**b22**
	3	4

Table 2. Available Resources

Product 1	S11	S12	S13	S14	S15	S16	S17	S18	S19	S110	S111	S112
	65	60	50	50	58	60	65	67	80	80	65	60
Product 2	**S21**	**S22**	**S23**	**S24**	**S25**	**S26**	**S27**	**S28**	**S29**	**S210**	**S211**	**S212**
	90	85	85	85	88	85	90	93	109	109	95	93

Table 4. Inventory Handling Costs.

Product 1	h11	h12	h13	h14	h15	h16	h17	h18	h19	h110	h111	h112
	40	40	40	40	40	40	40	40	40	40	40	40
Product 2	**h21**	**h22**	**h23**	**h24**	**h25**	**h26**	**h27**	**h28**	**h29**	**h210**	**h211**	**h212**
	40	40	40	40	40	40	40	40	40	40	40	40

Table 5. Inventory Costs

Inventory Cost Product 1	5
Inventory Cost Product 2	7

Table 6. Available labor resources.

R1	R2	R3	R4	R5	R6	R7	R8	R9	R10	R11	R12
151	150	146	150	155	145	165	170	187	185	170	160

Table 7. Demand.

Product 1	D11	D12	D13	D14	D15	D16	D17	D18	D19	D110	D111	D112
	10	15	12	14	10	13	12	20	13	15	17	12
Product 2	**D21**	**D22**	**D23**	**D24**	**D25**	**D26**	**D27**	**D28**	**D29**	**D210**	**D211**	**D212**
	10	15	12	12	14	10	13	14	15	12	15	13

Table 8. Minimum Demand.

	d11	d12	d13	d14	d15	d16	d17	d18	d19	d110	d111	d112
Product 1	10	10	12	10	13	13	14	16	15	16	12	10
	d21	**d22**	**d23**	**d24**	**d25**	**d26**	**d27**	**d28**	**d29**	**d210**	**d211**	**d212**
Product 2	13	12	12	12	10	10	15	10	16	14	13	12

Table 9. Available Capacity

Product 1	C11	C12	C13	C14	C15	C16	C17	C18	C19	C110	C111	C112
	13	13	12	15	14	15	13	18	15	17	14	12
Product 2	**C21**	**C22**	**C23**	**C24**	**C25**	**C26**	**C27**	**C28**	**C29**	**C210**	**C211**	**C212**
	13	13	12	14	13	15	17	14	17	14	13	12

The demand (Dij) is given in units; the yields (cij), inventory costs (hij), are given in monetary units; The capacity (γij) is given in units, and considering these parameters, each row, and each column in table 1 represent the products and periods, respectively.

Resource availability (Skj) is given in quantity of components k available to be used in each period j, where each row represents a component type and each column represent the periods. The bill of material (bki) is given in quantity components of type k needed to produce one unit of product i, and for this case, each row in table 1 represents a component type and each column represents a product.

Labor (Ri) (measured in standard time) is given in minutes, where each column represents a product. And labor availability (measured in standard time) (Rj) is also given in minutes, where each column represents a period.

5.2 Results

Tables 10 and 11 show the results for products 1 and 2 respectively, these optimal solutions are for the production planning model (2) without the decomposition.

Table 10. Optimal solution result for product 1 without decomposition.

Results for product 1 in the 12 periods					
x_{11}	x_{12}	x_{13}	x_{14}	x_{15}	x_{16}
12	13	12	10	13	14
I_{11}	I_{12}	I_{13}	I_{14}	I_{15}	I_{16}

2	0	0	0	3	4
x_{17}	x_{18}	x_{19}	x_{110}	x_{111}	x_{112}
12	16	15	16	14	12
I_{17}	I_{18}	I_{19}	I_{110}	I_{111}	I_{112}
4	0	2	3	0	0

Table 11. Optimal solution result for product 2 without decomposition.

Results for product 2 in the 12 periods					
x_{21}	x_{22}	x_{23}	x_{24}	x_{25}	x_{26}
13	12	12	14	12	10
I_{21}	I_{22}	I_{23}	I_{24}	I_{25}	I_{26}
3	0	0	2	0	0
x_{27}	x_{28}	x_{29}	x_{210}	x_{211}	x_{212}
15	12	16	14	13	12
I_{27}	I_{28}	I_{29}	I_{210}	I_{211}	I_{212}
2	0	1	3	1	0

And for each 1 of the 12 periods the value of the optimal solution was as follows:

Table 12. Result of the optimal solution value.

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
R$ 680,00	R$ 661,00	R$ 669,00	R$ 628,00	R$ 720,00	R$ 500,00	R$ 576,00	R$ 568,00	R$ 545,00	R$ 506,00	R$ 648,00	R$ 636,00

Having a total of z= R$ 7,337.00.

For the production planning problem solved through the decomposition scheme of the proposed mathematical model, the input data are the same as presented in tables 1-9, because both the general problem and the problem solved period by period receive the same input information, however, with the planning decomposition method we can decrease the available resources of labor and components (BOM) along the periods. Tables 13 and 14 show the solutions to this problem solved period by period.

Table 13. Result of the optimal solution for product 1 with the decomposition

Results for product 1 in the 12 periods					
x_{11}	x_{12}	x_{13}	x_{14}	x_{15}	x_{16}
12	13	12	10	13	14
I_{11}	I_{12}	I_{13}	I_{14}	I_{15}	I_{16}
2	0	0	0	3	4
x_{17}	x_{18}	x_{19}	x_{110}	x_{111}	x_{112}
12	16	15	16	14	12
I_{17}	I_{18}	I_{19}	I_{110}	I_{111}	I_{112}
4	0	2	3	0	0

Table 14. Result of the optimal solution for product 2 with the decomposition

Results for product 2 in the 12 periods					
x_{21}	x_{22}	x_{23}	x_{24}	x_{25}	x_{26}
13	12	12	14	12	10
I_{21}	I_{22}	I_{23}	I_{24}	I_{25}	I_{26}
3	0	0	2	0	0
x_{27}	x_{28}	x_{29}	x_{210}	x_{211}	x_{212}
15	12	16	14	13	12
I_{27}	I_{28}	I_{29}	I_{210}	I_{211}	I_{212}
2	0	1	3	1	0

Tables 13 and 14 present the results for the decomposition scheme of the proposed mathematical model discussed in this paper, in which it preserves the optimal solution present in tables 11 and 12 for the model without the decomposition having the same optimal value of z= R$7,337.00. However, as mentioned previously, this is the optimal solution, because there are savings in resource and computational expenses. Given that, the result of the fundamental problem is that we propose a decomposition scheme that reduces rework to the minimum possible, since it only plans the number of periods to be executed.

It is important to clarify that the planning horizon can be divided into any number M of blocks that the value of the optimal solution obtained will be the same as we would have obtained if all new data were incorporated at the beginning of the planning horizon, and that the problem was solved without decomposition. The decomposition scheme of the mathematical model proposed in this work is an improvement of the models presented in [6] and [5]. This scheme was modified to include the inventory level and uncertainty in the input parameters, such as demand, production capacity, production costs, labor resources, so that planning in a dynamic environment is in line with factory reality. In addition, to enunciate a theorem that ensures that the new proposed scheme given by the two-phase algorithm executes correctly in the new mathematical model (2). The scheme is quite flexible for planning parameters such as demand, production capacity, production resources, and others, and the proposed decomposition assumes that the planning horizon is fixed, and it is shortening over time, which is customary in cases of concessions of public services to individuals.

In summary, Tables 13 and 14 illustrate the role of the new two-phase algorithm using a numerical example. Furthermore, the algorithm seems to perform well in dynamic planning environments and allows for data update at the beginning of each period block over the planning horizon. These results are relevant in view of the fact that the new approach ensures not only that linear programming models coupled to CF can be decomposed properly, but also that the model can be updated along with the planning horizon. Finally, it is important to say that in the dynamic environment in which it provides a large reduction in planning costs for these companies and waste of resources over the horizon.

6 Final Considerations and Conclusions

The work presented addressed the problem of production planning in dynamic environments, where the planning parameters are subject to change throughout the planning horizon. In these cases, there are at least two challenges: keeping the planning up to date, for the good functioning and result of the model, and reducing costs with replanning, which was the objective of the work. The study focused on the issue of using Linear Programming for production planning in dynamic environments, and with long planning horizons. The classical schemes presented throughout the paper, which are used to decompose the problem resulting from the coupling of the Clearing Function to the linear programming model, do not work properly in the presence of capacity constraints. Since the Clearing Function is a capacity constraint, the classical decomposition scheme does not work [16].

The decomposition scheme of the mathematical model proposed to solve the deficiencies of the existing models is based on a decomposition of the planning model initially proposed by [2], and extended by [18],[5], [17], and contemplates the need to update the information that is made available throughout the planning horizon, thus avoiding rework, which for any productive system means loss of resources and consequently, of money.

The longer the planning horizon and the greater the dynamics of the environment, the greater the uncertainty of the input parameters in the planning system, which moves the initial planning further and further away from the reality of future periods to be covered by planning. Which brings us to the problem addressed in this work, where the number of planned periods is much larger than the number of periods to be executed. The decomposition method presented showed that it is possible to get around this problem by running the model period by period, or by blocks of periods, and in any case providing a lower cost solution to the problem of production planning in dynamic environments.

The analysis of the presented scheme shows that, in fact, the proposed decomposition scheme works well for the production planning problem because the decomposition algorithm receives the same information as the original problem and as both have the same solving rules. Therefore, it is reasonable to expect that the solution obtained from the decomposition is the same as the one obtained in the absence of the decomposition. We can then conclude that the greater the variability in the planning environment, and therefore in the planning parameters, the greater the resource savings, and the lower the planning distortions, when compared to a more stable environment. Furthermore, the results obtained are quite general since no simplifying assumptions were used, neither in the theorem nor in the rules of use of the algorithm.

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